Transformata Z

Pentru o variabilă complexă $z = x + jy = re^{j\Omega}$ transformata Z a semnalului discret x[n] este: $Z\{x[n]\}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$

Transformata Z evaluată pe un cerc unitar este chiar transformata Fourier discretă:

$$Z\{x[n]\}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = F\{x[n]\}(\Omega)$$

Pentru un SLIT, dacă $x[n] = \sum_k c_k z_k^{\,n} \Rightarrow y[n] = \sum_k c_k z_k^{\,n} H(z_k)$

Domeniul de convergență = coroana circulară centrată pe origine care nu conține poli ai transformatei. **Proprietăti DC**

1. pt semnale cu întindere spre dreapta, DC = coroana circulară delimitată de raza celui mai mare pol până la infinit (<u>semnal cauzal</u>)
2. pt semnale cu întindere spre stânga, DC = disc delimitat de raza celui mai mic pol (<u>semnal necauzal</u>)
3. pt semnale cu întindere de la $-\infty$ la ∞, DC este o coroana circulară ce nu include poli
4. semnalele cu suport mărginit au transformata Z definita in tot planul, cu exceptia z=0 si z=∞.
5. <u>sistemul stabil</u> <=> polii sunt în interiorul cercului unitate. DC se află în exteriorul cercului unitate.

<u>Probleme</u>

1. a) Să se demonstreze că pentru o secvență pară x[n]=x[-n] este adevarată egalitatea X(z)=X(1/z).

b) să se arate că polii (zerourile) acestei transformate Z sunt în perechi z_0, $1/z_0$.

c) demonstrați că semnalul în timp discret $x[n] = a^{|n|}$ este o secvență pară. Să se reprezinte grafic semnalul pentru a=3/4 si |n|<4.

d) determinați transformata Z a semnalului de la pct. c) şi domeniul de convergență.

e) găsiti polii şi zerourile acestei transformate.

2. Determinați transformatele z ale următoarelor semnale în timp discret. Să se precizeze în fiecare caz regiunea de convergență şi să se deseneze constelația de poli şi zerouri. Să se precizeze dacă aceste secvențe admit sau nu transformata Fourier în timp discret.

a) $\delta[n]$, b) $0.5^n \sigma[n]$,

c) $0.5^n \sigma[-n]$, d) $0.5^{|n|}$,

e) $0.5\{\sigma[n] - \sigma[n-2]\}$, f) $\sin(\Omega_0 n)\sigma[n]$, g) $a^n \sin(\Omega_0 n)\sigma[n]$.

3. Ştiind că X(z) reprezintă transformata Z a secvenței x[n] să se calculeze în funcție de X(z) transformata Z a următoarelor secvențe:

a) $\Delta\{x[n]\} = x[n] - x[n-1]$

b) $x_1[n] = (-1)^n x[n]$

c) $x_2[n] = \begin{cases} x\left[\dfrac{n}{2}\right], & n - par \\ 0, & n - impar \end{cases}$

d) $x_3[n] = x[2n]$.

4. Să se determine semnalele în timp discret care corespund următoarelor transformate Z

a) $5\left(1 - z^{-1}\right)\left(1 + z^{-3}\right)$

b) $\dfrac{1}{\left(1 - az^{-1}\right)^2}$ pentru cele două domenii de convergență posibile.

c) $\dfrac{2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)^2}$, pentru $\dfrac{1}{4} < |z| < \dfrac{1}{2}$

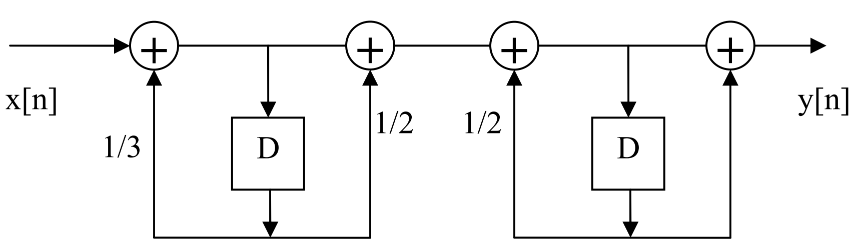
d) $\dfrac{z^2}{z^2 - 0.5z + 0.6}$

e) $\dfrac{1 + 3z^{-1} + \frac{11}{6}z^{-2} + \frac{1}{3}z^{-3}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$

f) $\dfrac{z\left(e^{-\alpha} - e^{-\beta}\right)}{\left(z - e^{-\alpha}\right)\left(z - e^{-\beta}\right)}$; α, β reale si pozitive.

g) $\log\left(1 + z^{-1}\right)$, pentru $|z| > |a|$

5. Pentru sistemul în timp discret din figură, considerat cauzal şi în repaus inițial



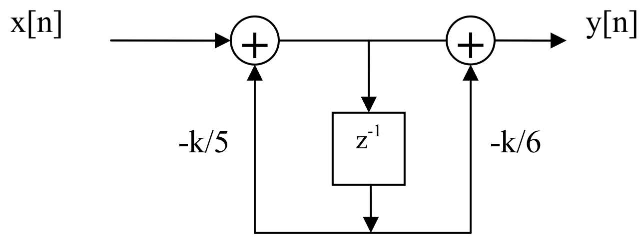
a) să se scrie ecuația cu diferențe finite
b) să se stabilească răspunsul la impuls h[n]
c) să se determine funcția de transfer H(z)
d) să se determine răspunsul în frecvență al sistemului
e) să se reprezinte grafic modulul şi argumentul funcției de la punctul d)

f) să se determine răspunsurile sistemului la următoarele semnale de intrare:

1. $\cos 10\pi n$; 2. $\left(\frac{1}{2}\right)^n \sigma[n]$; 3. $\sin^2(10\pi n)$; 4. $\sin(10\pi n)\cos(5\pi n)$; 5. $\sigma[n]$

g) să se deseneze forma canonică II de implementare a sistemului.

6. Se consideră sistemul în timp discret cauzal din figură:



a) să se determine funcția sa de transfer H(z). Schițați constelația de poli şi zerouri şi regiunea de convergență
b) pentru ce valori ale lui k sunt polii sistemului în interiorul cercului unitate?
c) să se determine y[n] dacă k=1 si $x[n]=\left(\frac{1}{2}\right)^{|n|}$

7. Un sistem liniar şi invariant în timp discret este descris de ecuația cu diferențe finite:

$$y[n]-2y[n-1]+y[n-2]=x[n]$$

a) să se determine funcția sa de transfer H(z). Schițați constelația de poli şi zerouri şi regiunea de convergență
b) să se determine răspunsul la impuls h[n]
c) să se analizeze stabilitatea sistemului.

Rezolvări.
Soluție problema 1

a) $x[n]=x[-n]$

$$x[-n]\xleftrightarrow{z}\sum_{n=-\infty}^{\infty}x[-n]z^{-n}\underline{\underline{m=-n}}\sum_{m=-\infty}^{\infty}x[m]z^{m}=\sum_{m=-\infty}^{\infty}x[m]\left(z^{-1}\right)^{-m}=X\left(z^{-1}\right)=X\left(\frac{1}{z}\right)$$

$$x[n]\leftrightarrow X(z)$$

Dar $\quad x[n]=x[-n]\Rightarrow X(z)=X\left(\frac{1}{z}\right)c.c.t.d.$

b) Fie z_0 un zero si z_p pol pentru $X(z)\Rightarrow$

$$X(z)=\frac{(z-z_0)P(z)}{(z-z_p)Q(z)} \Rightarrow X\left(\frac{1}{z}\right)=\frac{\left(\frac{1}{z}-z_0\right)P\left(\frac{1}{z}\right)}{\left(\frac{1}{z}-z_p\right)Q\left(\frac{1}{z}\right)}==\frac{(1-zz_0)P\left(\frac{1}{z}\right)}{(1-zz_p)Q\left(\frac{1}{z}\right)}=\frac{z_0}{z_p}\frac{\left(z-\frac{1}{z_0}\right)P\left(\frac{1}{z}\right)}{\left(z-\frac{1}{z_p}\right)Q\left(\frac{1}{z}\right)}$$

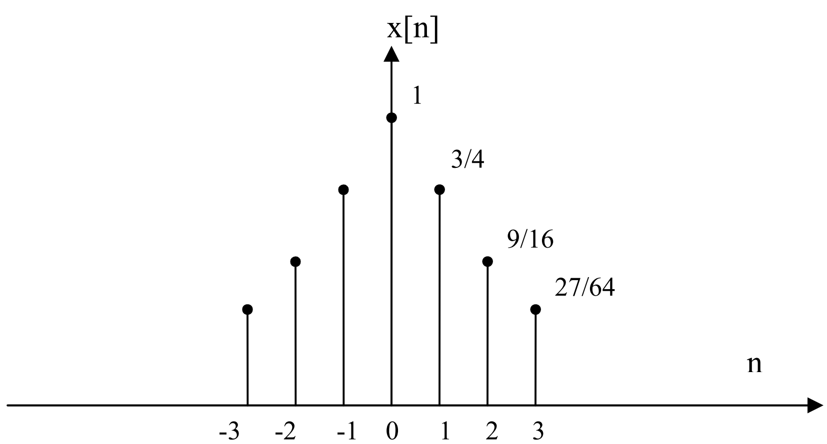
=> $\frac{1}{z_0}$ - zero si $\frac{1}{z_p}$ este pol al lui $X\left(\frac{1}{z}\right)$.

Dar $X\left(\frac{1}{z}\right)=X(z)$, deci $\frac{1}{z_0}$ - zero si $\frac{1}{z_p}$ este pol al lui $X(z)$.

Aşadar - zerourile lui $X(z)$ se pot grupa în perechi de forma $z_0, \frac{1}{z_0}$.

- polii lui $X(z)$ se pot grupa în perechi de forma $z_p, \frac{1}{z_p}$

c) $x[n]=a^{|n|}; x[-n]=a^{|-n|}=a^{|n|}=x[n] \Rightarrow x[n]$ este o funcție pară



d) $u[n]=\begin{Bmatrix} a^n, n\geq 0 \\ a^{-n}, n<0 \end{Bmatrix}=a^n\sigma[n]+a^{-n}\sigma[-n-1]=a^n\sigma[n]+\left(\frac{1}{a}\right)^n\sigma[-n-1]$

Dar : $a^n\sigma[n]\underset{|z|>|a|}{\leftrightarrow}\frac{1}{1-az^{-1}}$ şi $-b^n\sigma[-n-1]\underset{|z|<|b|}{\leftrightarrow}\frac{1}{1-bz^{-1}}$

Pentru $b=\frac{1}{a}\Rightarrow -\left(\frac{1}{a}\right)^n\sigma[-n-1]\underset{|z|<\frac{1}{|a|}}{\leftrightarrow}\frac{1}{1-\frac{1}{a}z^{-1}} \Leftrightarrow \left(\frac{1}{a}\right)^n\sigma[-n-1]\underset{|z|<\frac{1}{|a|}}{\leftrightarrow}\frac{-1}{1-\frac{1}{a}z^{-1}}$

Pentru $a=\frac{3}{4}$ regiunea de convergență este $\frac{3}{4}<|z|<\frac{4}{3}$ şi :

$$u[n]=\left(\frac{3}{4}\right)^{|n|}\leftrightarrow\frac{1}{1-\frac{3}{4}z^{-1}}-\frac{1}{1-\frac{4}{3}z^{-1}}=\frac{-\frac{7}{12}z^{-1}}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{4}{3}z^{-1}\right)}=U(z)$$

Polii lui $U(z)$ sunt $z_{p_1}=\frac{3}{4}$ şi $z_{p_2}=\frac{4}{3}$. Se observă că $z_{p_1}=\frac{1}{z_{p_2}}$ (1)

Zerourile lui $U(z)$ sunt $z_{0_1}=0$, si

$\lim\limits_{z\to\infty}U(z)=0\Rightarrow z_{0_2}=\infty$. Se observă că $z_{0_2}=\frac{1}{z_{0_1}}$ (2)

(1) şi (2) $\Rightarrow$ proprietatea de la ***b)*** este satisfăcută.

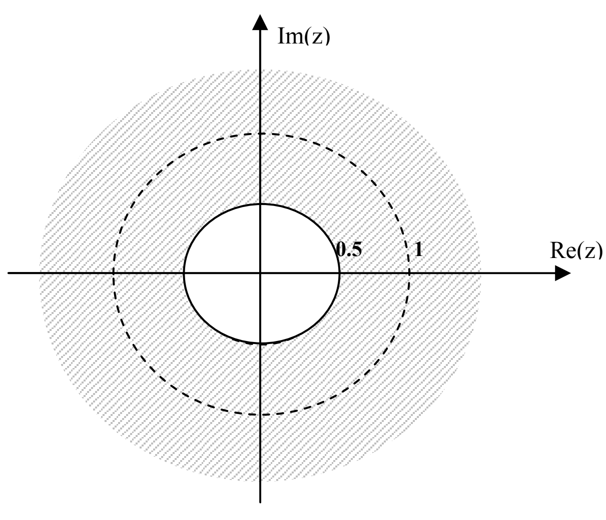
Soluție problema 2

a) $\delta[n]\leftrightarrow\sum\limits_{n=-\infty}^{\infty}\delta[n]z^{-n}=\delta[0]=1$ $\quad RDC=\{z\,|\,z\in\mathbb{C}\}$.

Transformata nu are nici poli nici zerouri. Secvența admite transformată Fourier în timp discret.

b) $0,5^n\sigma[n]\underset{|z|>0,5}{\leftrightarrow}\frac{1}{1-0,5z^{-1}}=\frac{z}{z-0,5}$

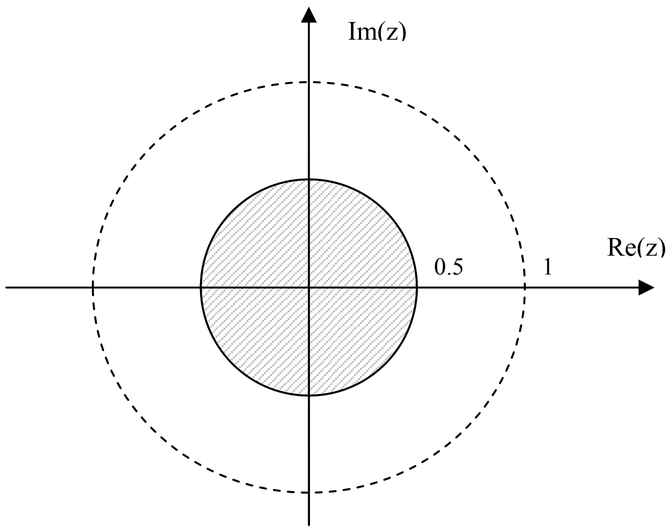
Secvența are transformată Fourier în timp discret.



c) $0,5^n\sigma[-n]\leftrightarrow\sum\limits_{n=-\infty}^{0}0,5^nz^n\overset{m=-n}{=}\sum\limits_{m=0}^{\infty}0,5^{-m}z^m=\sum\limits_{m=0}^{\infty}\left(\frac{z}{0,5}\right)^m$

Această serie de puteri este convergentă dacă $\left|\frac{z}{0,5}\right|<1 \Leftrightarrow |z|<0,5 \leftarrow R.D.C.$

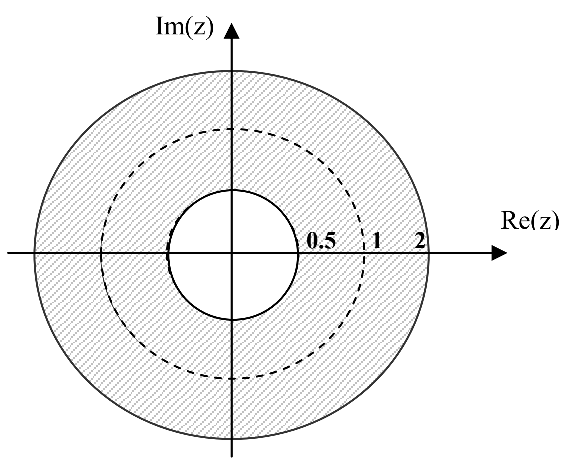
$0,5^n\sigma[-n] \leftrightarrow \frac{1}{1-\frac{z}{0,5}} = \frac{0,5}{0,5-z}$. Secvența **nu** are transformată Fourier în timp discret.



d) $0,5^{|n|} = \begin{Bmatrix} 0,5^n, n\geq 0 \\ 0,5^{-n}, n<0 \end{Bmatrix} = 0,5^n\sigma[n] + 0,5^{-n}\sigma[-n-1] = 0,5^n\sigma[n] + 2^n\sigma[-n-1]$

$0,5^n\sigma[n] \underset{|z|>0,5}{\leftrightarrow} \frac{1}{1-0,5z^{-1}}; 2^n\sigma[-n-1] \underset{|z|<2}{\leftrightarrow} -\frac{1}{1-2z^{-1}} \Rightarrow R.D.C. = \left\{z\in\mathbb{C} \middle| 0,5<|z|<2\right\}$

$0,5^{|n|} \leftrightarrow \frac{1}{1-0,5z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-1,5z^{-1}}{\left(1-0,5z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{1,5z}{(z-0,5)(z-2)}$



Secvența are transformată Fourier în timp discret.

e) $0,5\{\sigma[n]-\sigma[n-2]\}=0,5\{\delta[n]+\delta[n-1]\}\leftrightarrow 0,5+0,5z^{-1}=\dfrac{0,5(1+z)}{z}$

$R.D.C.=\{z\in\mathbb{C}\}-\{0\}$. Secvența are transformată Fourier în timp discret.

f) $\sin\Omega_0 n=\dfrac{e^{j\Omega_0 n}-e^{-j\Omega_0 n}}{2j}$

$$\sin\Omega_0 n\sigma[n]\leftrightarrow\frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j\Omega_0 n}-e^{-j\Omega_0 n}\right)z^{-n}=\frac{1}{2j}\left[\underbrace{\sum_{n=0}^{\infty}\left(\frac{e^{j\Omega_0}}{z}\right)^n}_{S_1}-\underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{e^{j\Omega_0}z}\right)^n}_{S_2}\right]$$

Seria de puteri S_1 este convergentă dacă $\left|\dfrac{e^{j\Omega_0}}{z}\right|<1\Leftrightarrow\dfrac{1}{|z|}<1\Leftrightarrow|z|>1$

Seria de puteri S_2 este convergentă dacă $\left|\dfrac{1}{e^{j\Omega_0}z}\right|<1\Leftrightarrow\dfrac{1}{|z|}<1\Leftrightarrow|z|>1$

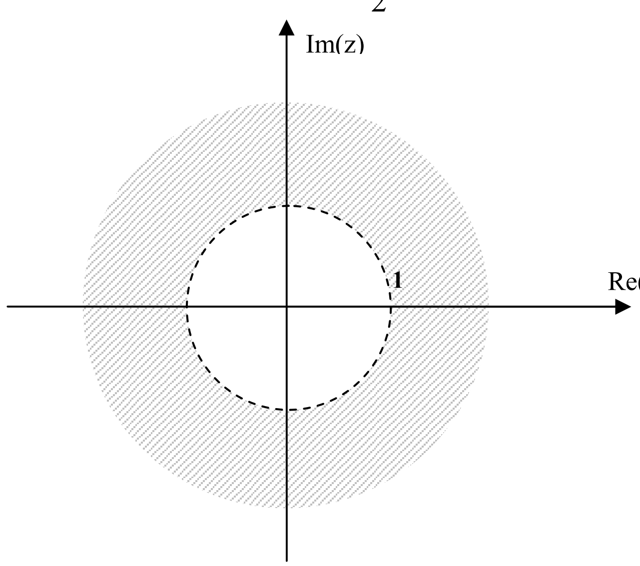
$R.D.C.=\left\{z\in\mathbb{C}\,\middle|\,|z|>1\right\}$. Secvența **nu** are transformată Fourier în timp discret.

$$\sin\Omega_0 n\sigma[n]\leftrightarrow\frac{1}{2j}\left[\frac{1}{1-\frac{e^{j\Omega_0}}{z}}-\frac{1}{1-\frac{1}{e^{j\Omega_0}z}}\right]=\frac{1}{2j}\left[\frac{1}{1-e^{j\Omega_0}z^{-1}}-\frac{1}{1-e^{-j\Omega_0}z^{-1}}\right]$$

$$=\frac{e^{j\Omega_0}-e^{-j\Omega_0}}{2j}\,\frac{z^{-1}}{1-e^{j\Omega_0}z^{-1}-e^{-j\Omega_0}z^{-1}+z^{-2}}=\sin\Omega_0\,\frac{z^{-1}}{1-2\cos\Omega_0 z^{-1}+z^{-2}}$$

Deci : $\sin\Omega_0 n\sigma[n]\leftrightarrow\sin\Omega_0\,\dfrac{z^{-1}}{1-2\cos\Omega_0 z^{-1}+z^{-2}}=\sin\Omega_0\,\dfrac{z}{z^2-2\cos\Omega_0 z+1}$

$$z^2-2\cos\Omega_0 z+1=0\Rightarrow z_{p_1,p_2}=\frac{\sqrt{4\cos^2\Omega_0-4}}{2}=\cos\Omega_0\pm j\sin\Omega_0=e^{\pm j\Omega_0}$$



g) $a^n \sin\Omega_0 n = \frac{a^n}{2j}\left(e^{j\Omega_0 n} - e^{-j\Omega_0 n}\right)$

$$a^n \sin\Omega_0 n\sigma[n] \leftrightarrow \frac{1}{2j}\left(\sum_{n=0}^{\infty} a^n e^{j\Omega_0 n} z^{-n} - \sum_{n=0}^{\infty} a^n e^{-j\Omega_0 n} z^{-n}\right) = \frac{1}{2j}\left(\underbrace{\sum_{n=0}^{\infty}\left(\frac{ae^{j\Omega_0}}{z}\right)^n}_{S_1} - \underbrace{\sum_{n=0}^{\infty}\left(\frac{ae^{-j\Omega_0}}{z}\right)^n}_{S_2}\right)$$

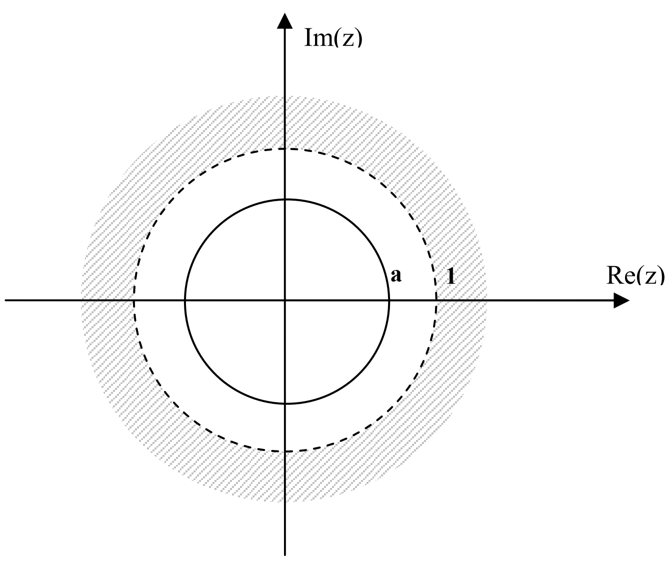
Seria de puteri S_1 este convergentă dacă $\left|\frac{ae^{j\Omega_0}}{z}\right| < 1 \Leftrightarrow \frac{|a|}{|z|} < 1 \Leftrightarrow |z| > |a|$

Seria de puteri S_2 este convergentă dacă $\left|\frac{ae^{-j\Omega_0}}{z}\right| < 1 \Leftrightarrow \frac{|a|}{|z|} < 1 \Leftrightarrow |z| > |a|$

$$R.D.C. = \left\{z \in \mathbb{C} \,\middle|\, |z| > |a|\right\}$$

$$a^n \sin\Omega_0 n\sigma[n] \leftrightarrow \frac{1}{2j}\left(\frac{1}{1-\frac{ae^{j\Omega_0}}{z}} - \frac{1}{1-\frac{ae-j\Omega_0}{z}}\right) = \frac{az\sin\Omega_0}{z^2 - 2a\cos\Omega_0 z + a^2}$$

$$z_{p_1,p_2} = \frac{2a\cos\Omega_0 \pm \sqrt{4a^2\cos^2\Omega_0 - 4a^2}}{2} = a\cos\Omega_0 \pm ja\sin\Omega_0 = ae^{\pm j\Omega_0}$$



Secvența are transformată Fourier în timp discret dacă $|a| < 1$.

Soluție problema 3.

a) $x[n] \leftrightarrow X(z)$; $x[n-1] \leftrightarrow z^{-1}X(z)$

$$\Delta\{x[n]\} = x[n] - x[n-1] \leftrightarrow \left(1 - z^{-1}\right)X(z)$$

b) $x_1[n]=(-1)^n x[n] \leftrightarrow \sum(-1)^n x[n]z^{-n}=\sum x[n](-z)^n = X(-z)$

c) $x_2[n]=\begin{cases} x\left[\frac{n}{2}\right], & n-par \\ 0, & n-impar \end{cases}$

$$X_2(z)=\sum_n x_2[n]z^{-n}=\sum_p x_2[2p]z^{-2p}+\sum_p x_2[2p+1]z^{-(2p+1)}=\sum_p x[p]z^{-2p}=X(z^2)$$

d) $x_3[n]=x[2n] \leftrightarrow X_3(z)=\sum x[2n]z^{-n}$

$$X(z)=\sum x[n]z^{-n}=\sum_p x[2p]z^{-2p}+\sum_p x[2p+1]z^{-(2p+1)}$$

$$X(-z)=\sum x[n](-z)^{-n}=\sum_p x[2p](-z)^{-2p}+\sum_p x[2p+1](-z)^{-(2p+1)}$$

$$X(-z)=\sum_p x[2p]z^{-2p}-\sum_p x[2p+1]z^{-(2p+1)}$$

$$=> \frac{1}{2}\left(X(z)+X(-z)\right)=\sum_p x[2p]z^{-2p} \;=> \frac{1}{2}\left(X(u^{\frac{1}{2}})+X(-u^{\frac{1}{2}})\right)=\sum_p x[2p]u^{-p}$$

$$X_3(z)=\frac{1}{2}\left(X(z^{\frac{1}{2}})+X(-z^{\frac{1}{2}})\right)$$

Soluție problema 4.

a) $X(z)=5\left(1-z^{-1}\right)\left(1+z^{-3}\right)=5-5z^{-1}+5z^{-3}-5z^{-4}$

$x[n]=5\delta[n]-5\delta[n-1]+5\delta[n-3]-5\delta[n-4]$

b)

$$X(z)=\frac{1}{\left(1-az^{-1}\right)^2};\text{Fie } X_1(z)=\frac{1}{1-az^{-1}} \Rightarrow \frac{dX_1(z)}{dz}=\frac{d}{dz}\left(\frac{z}{z-a}\right)=\frac{z-a-z}{(z-a)^2}$$

$$\Rightarrow -\frac{dX_1(z)}{dz}=\frac{a}{(z-a)^2} \Rightarrow -z^2\frac{dX_1(z)}{dz}=\frac{az^2}{(z-a)^2}=\frac{a}{\left(1-az^{-1}\right)^2}$$

$$\Rightarrow X(z)=\frac{1}{a}\left[-z^2\frac{dX_1(z)}{dz}\right]=-\frac{z^2}{a}\frac{dX_1(z)}{dz}$$

Dar: $X_1(z)=\dfrac{1}{1-az^{-1}} \leftrightarrow \begin{cases} a^n\sigma[n], |z|>|a| \\ -a^n\sigma^n[-n-1], |z|<|a| \end{cases}$

Cazul I : $|z|>|a| \Rightarrow x_1[n]=a^n\sigma[n]$

$$-z\frac{dX_1(z)}{dz} \leftrightarrow nx_1[n] \Rightarrow +\frac{z}{a}\left(-z\frac{dX_1(z)}{dz}\right) \leftrightarrow \frac{1}{a}(n+1)x_1[n+1]$$

$$\Rightarrow x[n]=\frac{1}{a}(n+1)a^{n+1}\sigma[n+1] \Leftrightarrow x[n]=(n+1)a^n\sigma[n+1]$$

Cazul II

$$|z|<|a| \Rightarrow x_1[n]=-a^n\sigma[-n-1] \Rightarrow x[n]=\frac{1}{a}(n+1)\left[-a^{n+1}\sigma[-n-2]\right]$$

$$x[n]=-(n+1)a^n\sigma[-n-2]$$

c) $$X(z)=\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)^2}=\frac{A_1}{1+\frac{1}{4}z^{-1}}+\frac{A_2}{\left(1+\frac{1}{4}z^{-1}\right)^2}+\frac{B}{1-\frac{1}{2}z^{-1}}$$

$$B=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z^{-1}=2}=\frac{2}{\left(1+\frac{1}{4}z^{-1}\right)^2}\Big|_{z^{-1}=2}=\frac{8}{9}$$

$$A_2=\left(1+\frac{1}{4}z^{-1}\right)^2X(z)\Big|_{z^{-1}=-4}=\frac{2}{1-\frac{1}{2}z^{-1}}\Big|_{z^{-1}=-4}=\frac{2}{3}\,;$$

$$\frac{d}{dz}\left[\left(1+\frac{1}{4}z^{-1}\right)^2X(z)\right]\Big|_{z^{-1}=-4}=A_1\frac{d}{dz}\left[1+\frac{1}{4}z^{-1}\right]\Big|_{z^{-1}=-4}$$

$$\Leftrightarrow \frac{d}{dz}\left[\frac{2}{1-\frac{1}{2}z^{-1}}\right]\Big|_{z^{-1}=-4}=-A_1\frac{1}{4}z^{-2}\Big|_{z^{-1}=-4} \Leftrightarrow -2\left(1-\frac{1}{2}z^{-1}\right)^{-2}\frac{z^{-2}}{2}\Big|_{z^{-1}=-4}=-4A_1 \Leftrightarrow A_1=\frac{4}{9}$$

$$X(z)=\frac{4}{9}\frac{1}{1+\frac{1}{4}z^{-1}}+\frac{2}{3}\frac{1}{\left(1+\frac{1}{4}z^{-1}\right)^2}+\frac{8}{9}\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$x[n]=\frac{4}{9}\left(\frac{1}{4}\right)^n\sigma[n]+\frac{2}{3}(n+1)a^n\sigma[n]-\frac{8}{9}\left(\frac{1}{2}\right)^n\sigma[-n-1]$$

d) $X(z)=\dfrac{10}{10-5z^{-1}+6z^{-2}}$ $\quad z_{p_1,p_2}^{-1}=\dfrac{5\pm\sqrt{25-240}}{12}=\dfrac{5\pm j\sqrt{215}}{12}$ $\quad \left|z_{p_1}{}^{-1}\right|=\left|z_{p_2}{}^{-1}\right|=m$

$$X(z)=\frac{A}{z^{-1}-z_{p_1}{}^{-1}}+\frac{B}{z^{-1}-z_{p_2}{}^{-1}};$$

$$A=\left(z^{-1}-z_{p_1}^{-1}\right)X\left(z\right)\Big|_{z^{-1}=z_{p_1}^{-1}};B=\left(z^{-1}-z_{p_1}^{-1}\right)X\left(z\right)\Big|_{z^{-1}=z_{p_1}^{-1}}$$

$$X\left(z\right)=\frac{\frac{A}{z_{p_1}^{-1}}}{\frac{z^{-1}}{z_{p_1}^{-1}}-1}+\frac{\frac{B}{z_{p_2}^{-1}}}{\frac{z^{-1}}{z_{p_2}^{-1}}-1}$$

Caz I $\quad |z|>m \quad x[n]=-\frac{A}{z_{p_1}^{-1}}\left(\frac{1}{z_{p_1}^{-1}}\right)^n\sigma[n]-\frac{B}{z_{p_2}^{-1}}\left(\frac{1}{z_{p_2}^{-1}}\right)\sigma[n]$

Caz II $\quad |z|<m \quad x[n]=\frac{A}{z_{p_1}^{-1}}\left(\frac{1}{z_{p_1}^{-1}}\right)^n\sigma[-n-1]+\frac{B}{z_{p_2}^{-1}}\left(\frac{1}{z_{p_2}^{-1}}\right)^n\sigma[-n-1]$

e)

$$X\left(z\right)=\frac{\frac{1}{3}z^{-3}+\frac{11}{6}z^{-2}+3z^{-1}+1}{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1}=2\frac{\frac{1}{6}z^{-3}+\frac{5}{6}z^{-2}+z^{-1}+\frac{1}{12}z^{-2}+\frac{1}{2}z^{-1}+\frac{1}{2}}{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1}$$

$$=2z^{-1}+\frac{z^{-2}+z^{-1}+1}{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1}=2z^{-1}+6\frac{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1-\frac{4}{6}z^{-1}-\frac{5}{6}}{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1}$$

$$=2z^{-1}+6-\underbrace{\frac{4z^{-1}+5}{\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1}}_{X_1(z)}$$

$$X_1\left(z\right)=\frac{24z^{-1}+30}{z^{-2}+5z^{-1}+6}=\frac{24z^{-1}+30}{\left(z^{-1}+2\right)\left(z^{-1}+3\right)}$$

$$X_1\left(z\right)=\frac{A}{z^{-1}+2}+\frac{B}{z^{-1}+3};A=\left(z^{-1}+2\right)X_1\left(z\right)\Big|_{z^{-1}=-2}=\frac{24z^{-1}+30}{z^{-1}+3}\Big|_{z^{-1}=-2}=-18$$

$$B=\left(z^{-1}+3\right)X_1\left(z\right)\Big|_{z^{-1}=-3}=\frac{24z^{-1}+30}{z^{-1}+2}\Big|_{z^{-1}=-3}=-42$$

$$X_1\left(z\right)=-18\frac{1}{z^{-1}+2}+42\frac{1}{z^{-1}+3}=-9\frac{1}{1+\left(\frac{1}{2}\right)z^{-1}}+14\frac{1}{1+\left(\frac{1}{3}\right)z^{-1}}$$

Caz I $\quad |z|>\frac{1}{2};$

$$x[n]=2\delta[n-1]+6\delta[n]-\left(-9\left(-\frac{1}{2}\right)^n\sigma[n]+14\left(-\frac{1}{3}\right)^n\sigma[n]\right)$$

$$x[n]=2\delta[n-1]+6\delta[n]+9\left(-\frac{1}{2}\right)^n\sigma[n]-14\left(-\frac{1}{3}\right)^n\sigma[n]$$

Caz II $\frac{1}{3}<|z|<\frac{1}{2}; x[n]=2\delta[n-1]+6\delta[n]-9\left(-\frac{1}{2}\right)^n\sigma[-n-1]+14\left(-\frac{1}{3}\right)^n\sigma[n]$

Caz III $|z|<\frac{1}{3}; x[n]=2\delta[n-1]+6\delta[n]-\left(9\left(-\frac{1}{2}\right)^n\sigma[-n-1]\right)-14\left(-\frac{1}{3}\right)^n\sigma[-n-1]$**)**

$$X(z)=\frac{z\left(e^{-\alpha}-e^{-\beta}\right)}{\left(z-e^{-\alpha}\right)\left(z-e^{-\beta}\right)}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{\left(1-e^{-\alpha}z^{-1}\right)\left(1-e^{-\beta}z^{-1}\right)}=\frac{A}{1-e^{-\alpha}z^{-1}}+\frac{B}{1-e^{-\beta}z^{-1}}$$

$$A=\left(1-e^{-\alpha}z^{-1}\right)X(z)\Big|_{z^{-1}=e^{\alpha}}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\beta}z^{-1}}\Bigg|_{z^{-1}=e^{\alpha}}=\frac{e^{\alpha}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\beta}e^{\alpha}}=1$$

$$B=\left(1-e^{-\beta}z^{-1}\right)X(z)\Big|_{z^{-1}=e^{-\beta}}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\alpha}z^{-1}}\Bigg|_{z^{-1}=e^{\beta}}=\frac{e^{\beta-\alpha}-1}{1-e^{\beta-\alpha}}=-1$$

$$X(z)=\frac{1}{1-e^{-\alpha}z^{-1}}$$

$$X(z)=\frac{z\left(e^{-\alpha}-e^{-\beta}\right)}{\left(z-e^{-\alpha}\right)\left(z-e^{-\beta}\right)}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{\left(1-e^{-\alpha}z^{-1}\right)\left(1-e^{-\beta}z^{-1}\right)}=\frac{A}{1-e^{-\alpha}z^{-1}}+\frac{B}{1-e^{-\beta}z^{-1}}$$

$$A=\left(1-e^{-\alpha}z^{-1}\right)X(z)\Big|_{z^{-1}=e^{\alpha}}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\beta}z^{-1}}\Bigg|_{z^{-1}=e^{\alpha}}=\frac{e^{\alpha}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\beta}e^{\alpha}}=1$$

$$B=\left(1-e^{-\beta}z^{-1}\right)X(z)\Big|_{z^{-1}=e^{-\beta}}=\frac{z^{-1}\left(e^{-\alpha}-e^{-\beta}\right)}{1-e^{-\alpha}z^{-1}}\Bigg|_{z^{-1}=e^{\beta}}=\frac{e^{\beta-\alpha}-1}{1-e^{\beta-\alpha}}=-1$$

$$X(z)=\frac{1}{1-e^{-\alpha}z^{-1}}$$

Considerăm $\alpha<\beta$

Caz I $|z|>e^{-\alpha}>e^{-\beta}; x[n]=e^{-\alpha n}\sigma[n]-e^{-\beta n}\sigma[n]$

Caz II $e^{-\alpha}>|z|>e^{-\beta}; x[n]=-e^{-\alpha n}\sigma[-n-1]-e^{-\beta n}\sigma[n]$

Caz III $|z|<e^{-\beta}<e^{-\alpha}; x[n]-e^{-\alpha n}\sigma[-n-1]+e^{-\beta n}\sigma[-n-1]$

g) $X(z)=\log\left(1+z^{-1}\right);|z|>|a|$**;** $\quad z^{-1}=u\Rightarrow X(u)=\log(1+u)$

Se dezvoltă această funcție în serie Taylor în jurul originii.

$$X(u)=\log(1+u)\Big|_{u=0}+\frac{1}{1!}\left[\log(1+u)\right]'\Big|_{u=0}u+\frac{1}{2!}\left[\log(1+u)\right]''\Big|_{u=0}u^2+\ldots=$$

$$=0+\frac{1}{1!}\frac{1}{1+u}\Big|_{u=0}u+\frac{1}{2!}\left(-\frac{1}{(1+u)^2}\right)\Big|_{u=0}u^2+\frac{1}{3!}\left(\frac{2}{(1+u)^3}\right)\Big|_{u=0}u^3+\ldots=$$

$$=u-\frac{1}{2!}u^2+\frac{1}{3!}2u^3+\ldots=\sum_{k=2}^{\infty}\frac{1}{k!}(k-1)u^k(-1)^{k-1}\Rightarrow$$

$$\Rightarrow X(z)=\sum_{k=2}^{\infty}\frac{1}{k!}(k-1)z^{-k}(-1)^{k-1}\Rightarrow x[n]=(-1)^{n-1}\frac{n-1}{n!}\sigma[n-1]$$

Soluție problema 5

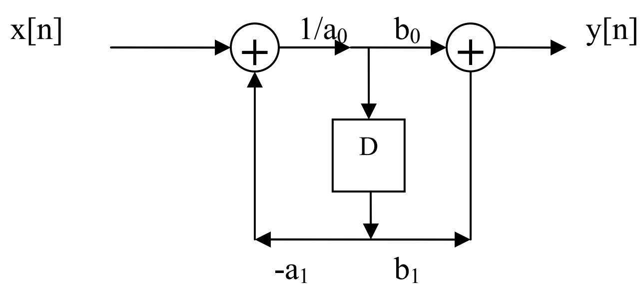
a) Un sistem de ordinul I este descris de ecuația cu diferențe finite:

$$a_0y[n]+a_1y[n-1]=b_0x[n]+b_1x[n-1]$$

şi de funcția de transfer:

$$H(z)=\frac{b_0+b_1z^{-1}}{a_0+a_1z^{-1}}$$

Forma canonică II de implementare a acestui sistem este:



Sistemul din enunțul acestei probleme este format prin conectarea în cascadă a două astfel de sisteme de ordinul I.

Pentru primul: $\frac{1}{a_1}=1; b_0=1; -a_1=\frac{1}{3}; b_1=\frac{1}{2}$

Funcția de transfer a primului sistem este: $H_1(z)=\dfrac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}$

Pentru al doilea sistem din enunț: $\frac{1}{a_1}=1; b_0=1; -a_1=\frac{1}{2}; b_1=1$

Funcția de transfer a sistem al doilea este: $H_2(z)=\dfrac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}$

Funcția de transfer a sistemului global este:

$$H(z)=H_1(z)\cdot H_2(z)=\frac{1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}{1-\left(\frac{1}{3}+\frac{1}{2}\right)z^{-1}+\frac{1}{6}z^{-2}}=\frac{1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

Coeficienții acestei funcții de transfer sunt: $b_0=1; b_1=\frac{3}{2}; a_0=1; a_1=-\frac{5}{6}; a_2=\frac{1}{6}$

Ecuația cu diferențe finite corespunzătoare este :

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=x[n]+\frac{3}{2}x[n-1]+\frac{1}{2}x[n-2]$$

b) $1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}=0 \Leftrightarrow z^{-2}-5z^{-1}+6=0; z_{1,2}=\frac{5\pm 1}{2}=\begin{cases}3\\2\end{cases}$

$$H(z)=\frac{3z^{-2}+9z^{-1}+6}{z^{-2}-5z^{-1}+6}=\frac{A}{z^{-1}-2}+\frac{B}{z^{-1}-3}$$

$$A=\left(z^{-1}-2\right)H(z)\Big|_{z^{-1}=2}=\frac{3z^{-2}+9z^{-1}+6}{z^{-1}-3}\Big|_{z^{-1}=2}=\frac{12+18+6}{-1}=-36$$

$$B=\left(z^{-1}-3\right)H(z)\Big|_{z^{-1}=3}=\frac{3z^{-2}+9z^{-1}+6}{z^{-1}-2}\Big|_{z^{-1}=3}=\frac{27+27+6}{1}=60$$

$$H(z)=-36\frac{1}{z^{-1}-2}+60\frac{1}{z^{-1}-3}=18\frac{1}{1-\frac{1}{2}z^{-1}}-20\frac{1}{1-\frac{1}{3}z^{-1}}$$

Dar:

$$\frac{1}{1-\frac{1}{2}z^{-1}}\overset{cauzal}{\leftrightarrow}\left(\frac{1}{2}\right)^n\sigma[n]$$

$$\frac{1}{1-\frac{1}{3}z^{-1}}\overset{cauzal}{\leftrightarrow}\left(\frac{1}{3}\right)^n\sigma[n]$$

=> Deci: $h[n]=18\left(\frac{1}{2}\right)^n\sigma[n]-20\left(\frac{1}{3}\right)^n\sigma[n]$

c) Funcția de transfer a fost deja determinată: $H(z)=\dfrac{3z^{-2}+9z^{-1}+6}{z^{-2}-5z^{-1}+6}$

d) $z=e^{j\Omega}\Rightarrow H(\Omega)=\dfrac{3e^{-2j\Omega}+9e^{-j\Omega}+6}{e^{-2j\Omega}-5e^{-j\Omega}+6}$

e)

$$H(\Omega)=\frac{3(\cos 2\Omega-j\sin 2\Omega)+9(\cos\Omega-j\sin\Omega)+6}{\cos 2\Omega-j\sin 2\Omega-5(\cos\Omega-j\sin\Omega)+6}=$$

$$\frac{3\cos 2\Omega+9\cos\Omega+6-j(3\sin 2\Omega+9\sin\Omega)}{\cos 2\Omega-5\cos\Omega+6-j(\sin 2\Omega-5\sin\Omega)}$$

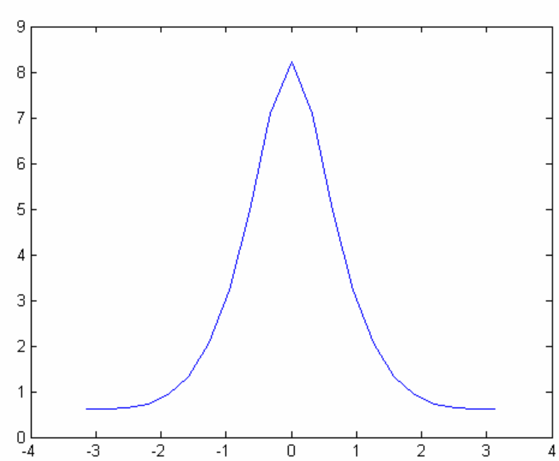
$$|H(\Omega)|=\sqrt{\frac{(3\cos 2\Omega+9\cos\Omega+6)^2+(3\sin 2\Omega+9\sin\Omega)^2}{(\cos 2\Omega-5\cos\Omega+6)^2+(\sin 2\Omega-5\sin\Omega)}}=$$

$$=\sqrt{\frac{126+54\cos\Omega+36\cos 2\Omega+108\cos\Omega}{62-10\cos\Omega+12\cos 2\Omega-60\cos\Omega}}=$$

$$=\sqrt{\frac{36(2\cos^2\Omega-1)+108\cos\Omega+126}{12(2\cos^2\Omega-1)-70\cos\Omega+62}}$$

$$|H(\Omega)|=\sqrt{\frac{72\cos^2\Omega+108\cos\Omega+90}{24\cos^2\Omega-70\cos\Omega+50}}$$

Ω	0	$\pi/4$	$\pi/2$	π
$\cos\Omega$	1	$\sqrt{2}/2$	0	-1
$\lvert H(\Omega)\rvert$	$3\sqrt{5/2}\cong 4,74$	$\sqrt{\dfrac{126+54\sqrt{2}}{62-35\sqrt{2}}}\cong 4,02$	$\sqrt{\dfrac{9}{5}}\cong 1,34$	$\sqrt{54}/12$ $\cong 0,61$



$$\arg\{H(\Omega)\} = \arg\{3\cos 2\Omega + 9\cos\Omega + 6 - j|3\sin 2\Omega + 9\sin\Omega|\} -$$
$$-\arg\{\cos 2\Omega - 5\cos\Omega + 6 - j(\sin 2\Omega - 5\sin\Omega)\}$$

Pentru $0 \le \Omega \le \pi$

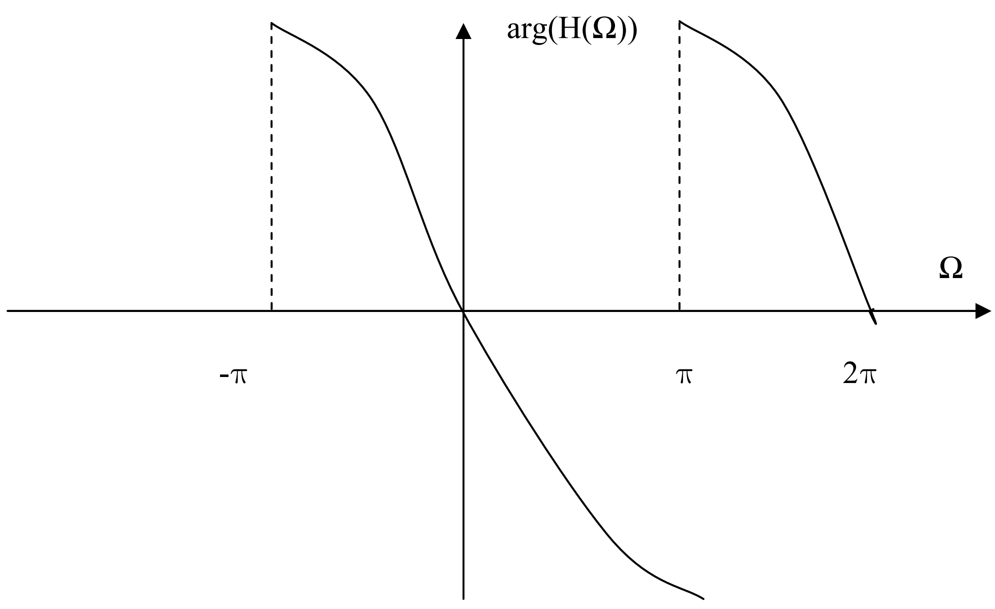
$$\arg\{H(\Omega)\} = -\mathrm{arctg}\frac{3\sin 2\Omega + 9\sin\Omega}{3\cos 2\Omega + 9\cos\Omega + 6} + \mathrm{arctg}\frac{\sin 2\Omega - 5\sin\Omega}{\cos 2\Omega - 5\cos\Omega + 6}$$

Dar: $$\frac{3\sin 2\Omega + 9\sin\Omega}{3\cos 2\Omega + 9\cos\Omega + 6} = \frac{3\sin\Omega(2\cos\Omega + 3)}{6\cos^2\Omega + 9\cos\Omega + 6} = \frac{\sin\Omega(2\cos\Omega + 3)}{2\cos^2\Omega + 3\cos\Omega + 1}$$

Şi : $$\frac{\sin 2\Omega - 5\sin\Omega}{\cos 2\Omega - 5\cos\Omega + 6} = \frac{\sin\Omega(2\cos\Omega - 5)}{2\cos^2\Omega - 5\cos\Omega + 5}$$

Deci : $$\arg\{H(\Omega)\} = -arctg\frac{\sin\Omega(2\cos\Omega + 3)}{2\cos^2\Omega + 3\cos\Omega + 1} + arctg\frac{\sin\Omega(2\cos\Omega - 5)}{2\cos^2\Omega - 5\cos\Omega + 5}$$

Ω	0	π/4	π/2	π
sinΩ	0	$1/\sqrt{2}$	1	0
cosΩ	1	$1/\sqrt{2}$	0	-1
Arg{H(Ω)}	0 °	-92°	-124°	-180°



f)1°)

$$x[n] = \cos 10\pi n \Rightarrow y[n] = |H(10\pi)|\cos(10\pi n + \arg\{H(10\pi)\})$$

$$|H(10\pi)| = |H(0 + 5\cdot 2\pi)| = |H(0)| = 4,74\,; \qquad \arg\{H(10\pi)\} = \arg\{H(0)\} = 0$$

$y[n] = 4,74\cos(10\pi n)$

2°) $x[n] = \left(\frac{1}{2}\right)^n \sigma[n] \Rightarrow X(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow Y(z) = X(z)H(z) \Rightarrow$

$$\Rightarrow Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}} \cdot \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{C}{1-\frac{1}{3}z^{-1}}$$

$$C = \left(1-\frac{1}{3}z^{-1}\right)Y(z)\Big|_{z^{-1}=3} = \frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)^2}\Bigg|_{z^{-1}=3} = \frac{\frac{5}{2}4}{\frac{1}{4}} = 40$$

$$B = \left(1-\frac{1}{2}z^{-1}\right)^2 Y(z)\Big|_{z^{-1}=2} = \frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}{1-\frac{1}{3}z^{-1}}\Bigg|_{z^{-1}=2} = \frac{6}{\frac{1}{3}} = 18$$

$$A = \frac{d}{dz}\left(\left(1-\frac{1}{2}z^{-1}\right)^2 Y(z)\right)\Bigg|_{z^{-1}=2} = \frac{d}{dz}\left(\frac{1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{1}{3}z^{-1}}\right)\Bigg|_{z^{-1}=2} = \frac{d}{dz}\left(\frac{z^2+\frac{3}{2}z+\frac{1}{2}}{z^2-\frac{1}{3}z}\right)\Bigg|_{z^{-1}=2} =$$

$$= \frac{\frac{7}{2}z^2+\frac{1}{6}}{\left(z^2-\frac{1}{3}z\right)^2}\Bigg|_{z^{-1}=2} = 150 \Rightarrow A = 150$$

Deci:

$$Y(z) = \underset{\substack{\updownarrow \\ 150\left(\frac{1}{2}\right)^n \sigma[n]}}{\frac{150}{1-\frac{1}{2}z^{-1}}} + \underset{\substack{\updownarrow \\ 40\left(\frac{1}{3}\right)^n \sigma[n]}}{\frac{18}{\left(1-\frac{1}{2}z^{-1}\right)^2}} \qquad \text{iar } \frac{18}{\left(1-\frac{1}{2}z^{-1}\right)^2} = 36\frac{\frac{1}{2}z \cdot z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

şi $\dfrac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \leftrightarrow n\left(\frac{1}{2}\right)^n \sigma[n]$, adică $36z\dfrac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \leftrightarrow 36(n+1)\left(\frac{1}{2}\right)^{n+1} \sigma[n+1]$

De aceea :

$$y[n]=150\left(\frac{1}{2}\right)^n\sigma[n]+36(n+1)\left(\frac{1}{2}\right)^{n+1}\sigma[n+1]+40\left(\frac{1}{3}\right)^n\sigma[n]$$

3°) $x[n]=\sin(10\pi n)=\dfrac{1-\cos(20\pi n)}{2}=0,\forall n\in\mathbb{Z}\Rightarrow y[n]=0,\forall n\in\mathbb{Z}$

4°) $\sin(10\pi n)\cos(5\pi n)=0,\forall n\in\mathbb{Z}\Rightarrow y[n]=0,(\forall)n\in\mathbb{Z}$

5°) $x[n]=\sigma[n]\Rightarrow X(z)\dfrac{1}{1-z^{-1}}\Rightarrow$

$$\Rightarrow Y(z)=X(z)\cdot H(z)=\frac{1}{1-z^{-1}}\cdot\frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}\cdot\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}=\frac{A}{1-z^{-1}}+\frac{B}{1-\frac{1}{3}z^{-1}}+\frac{C}{1-\frac{1}{2}z^{-1}}$$

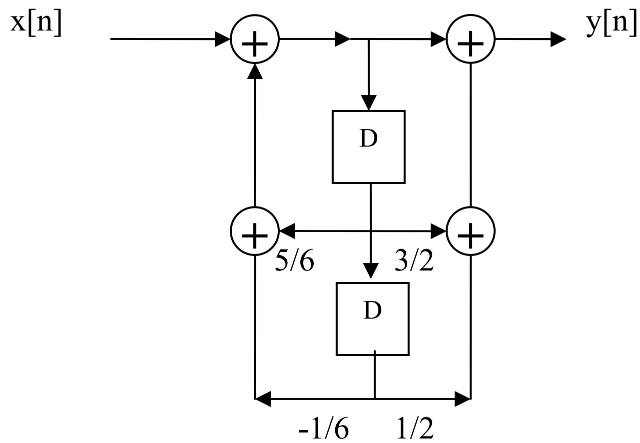
$$A=\left(-z^{-1}\right)Y(z)\Big|_{z^{-1}=1}=\frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}{\left(1-\frac{1}{3}z^{-1}\right)\left[1-\frac{1}{2}z^{-1}\right]}\Bigg|_{z^{-1}=1}=\frac{\frac{3}{2}\cdot 2}{\frac{2}{3}\cdot\frac{1}{2}}=9;$$

$$B=\left(1-\frac{1}{3}z^{-1}\right)Y(z)\Big|_{z^{-1}=3}=\frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\Bigg|_{z^{-1}=3}=\frac{\frac{5}{2}\cdot 4}{(-2)\left(-\frac{1}{2}\right)}=10$$

$$C=\left(1-\frac{1}{2}z^{-1}\right)Y(z)\Big|_{z^{-1}=2}=\frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}\Bigg|_{z^{-1}=2}=\frac{2\cdot 3}{(-1)\left(\frac{1}{3}\right)}=-18$$

$$Y(z)=\frac{9}{1-z^{-1}}+\frac{10}{1-\frac{1}{3}z^{-1}}+\frac{18}{1-\frac{1}{2}z^{-1}}\leftrightarrow y[n]=9\sigma[n]+10\left(\frac{1}{3}\right)^n\sigma[n]+18\left(\frac{1}{2}\right)^n\sigma[n]$$

g) $H(z)=\dfrac{1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}\Rightarrow a_0=1;a_1=-\dfrac{5}{6};a_2=\dfrac{1}{6};\qquad b_0=1;b_1=\dfrac{3}{2};b_2=\dfrac{1}{2}$



Solutie problema 6

a) $\frac{1}{a_0}=1; b_0=1; -a_1=-\frac{k}{5}; b_1=\frac{-k}{6}$.

$$H(z)=\frac{b_0+b_1z^{-1}}{a_0+a_1z^{-1}}=\frac{1-\frac{k}{6}z^{-1}}{1+\frac{k}{5}z^{-1}}=\frac{z-\frac{k}{6}}{z+\frac{k}{5}}\Rightarrow z_0=\frac{k}{6}; z_p=-\frac{k}{5}$$

b) $-1<-\frac{k}{5}<1\Rightarrow -5<k<5$

c) $x_1[n]=\left(\frac{1}{2}\right)^n=\left(\frac{1}{2}\right)^n\sigma[n]+\left(\frac{1}{2}\right)^n\sigma[-n-1]$

$$\left(\frac{1}{2}\right)^n\sigma[n]\overset{|z|>\frac{1}{2}}{\longleftrightarrow}\frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{şi} \qquad \left(\frac{1}{2}\right)^n\sigma[-n-1]\overset{|z|<\frac{1}{2}}{\longleftrightarrow}\frac{-1}{1-\frac{1}{2}z^{-1}}$$

Deci regiunea de convergență a transformatei z a semnalului $x_1[n]$ este mulțimea vidă.

$$x[n]=\left(\frac{1}{2}\right)^{|n|}=\begin{cases}\left(\frac{1}{2}\right)^n, n\geq 0\\ \left(\frac{1}{2}\right)^{-n}=2^n, n<0\end{cases}\Rightarrow x[n]=\left(\frac{1}{2}\right)^n\sigma[n]+2^n\sigma[-n-1]$$

$$\left(\frac{1}{2}\right)^n\sigma[n]\overset{|z|>\frac{1}{2}}{\longleftrightarrow}\frac{1}{1-\frac{1}{2}z^{-1}} \qquad \text{şi} \qquad 2^n\sigma[-n-1]\overset{|z|<2}{\longleftrightarrow}\frac{-1}{1-2z^{-1}}$$

Regiunea de convergență a transformatei z a semnalului $x[n]$ este $\left\{ z\in\mathbb{C} \left| \frac{1}{2}<|z|<2 \right. \right\}$

$$X(z)=\frac{1}{1-z^{-1}}-\frac{1}{1-2z^{-1}};$$

$$Y(z)=X(z)\cdot H(z)\overset{k=1}{\Rightarrow}Y(z)=\frac{1-\frac{1}{6}z^{-1}}{1+\frac{1}{5}z^{-1}}\left(\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-2z^{-1}}\right)$$

$$Y(z)=\underbrace{\frac{1-\frac{1}{6}z^{-1}}{\left(1+\frac{1}{5}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}}_{Y_2(z)}-\underbrace{\frac{1-\frac{1}{6}z^{-1}}{\left(1+\frac{1}{5}z^{-1}\right)\left(1-2z^{-1}\right)}}_{Y_2(z)}$$

$$Y_1(z)=\frac{A}{1+\frac{1}{5}z^{-1}}+\frac{B}{1-\frac{1}{2}z^{-1}};\ \Rightarrow A=\left(1+\frac{1}{5}z^{-1}\right)Y_1(z)\Big|_{z^{-1}=-5}=\frac{1-\frac{1}{6}z^{-1}}{1-\frac{1}{2}z^{-1}}\Bigg|_{z^{-1}=-5}=\frac{11}{21}$$

$$B=\left(1-\frac{1}{2}z^{-1}\right)Y_1(z)=\frac{1-\frac{1}{6}z^{-1}}{1+\frac{1}{5}z^{-1}}\Bigg|_{z^{-1}=2}=\frac{10}{21}$$

$$y_1[n]=\frac{11}{21}\left(-\frac{1}{5}\right)^n\sigma[n]+\frac{10}{21}\left(\frac{1}{2}\right)^n\sigma[n]$$

$$Y_2(z)=\frac{C}{1+\frac{1}{5}z^{-1}}+\frac{D}{1-2z^{-1}}\Rightarrow C=\left(1+\frac{1}{5}z^{-1}\right)Y_2(z)\Big|_{z^{-1}=-5}=\frac{1-\frac{1}{6}z^{-1}}{1-2z^{-1}}\Bigg|_{z^{-1}=-5}=\frac{1}{6}$$

$$D=\left(1-2z^{-1}\right)Y_2(z)\Bigg|_{z^{-1}=\frac{1}{2}}=\frac{1-\frac{1}{6}z^{-1}}{1+\frac{1}{5}z^{-1}}\Bigg|_{z^{-1}=\frac{1}{2}}=\frac{5}{6}$$

$$Y_2(z)=\frac{1}{6}\cdot\frac{1}{1+\frac{1}{5}z^{-1}}+\frac{5}{6}\cdot\frac{1}{1-2z^{-1}}\leftrightarrow y_2[n]=\frac{1}{6}\left(-\frac{1}{5}\right)^n\sigma[n]+\frac{5}{6}2^n\sigma[n]$$

$$y[n]=y_1[n]-y_2[n]=\frac{11}{21}\left(-\frac{1}{5}\right)^n\sigma[n]+\frac{10}{21}\left(\frac{1}{2}\right)^n\sigma[n]-\frac{1}{6}\left(-\frac{1}{5}\right)^n\sigma[n]-\frac{5}{6}2^n\sigma[n]$$

$$y[n]=\left(\frac{45}{126}\right)\left(-\frac{1}{5}\right)^n\sigma[n]+\frac{10}{21}\left(\frac{1}{2}\right)^n\sigma[n]-\frac{5}{6}2^n\sigma[n]$$

Solutie problema 7

a) $a_0=1; a_1=-2; a_2=1; b_0=1$ => $H(z)=\frac{1}{1-2z^{-1}+z^{-2}}=\frac{1}{\left(1-z^{-1}\right)^2}\Rightarrow z_{p_1}^{-1}=z_{p_2}^{-1}=1$

Există 2 regiuni de convergență : $|z|<1$ sau $|z|>1$.

b) $H_1(z)=\frac{1}{z^{-1}}\overset{|z|>1}{\leftrightarrow}\sigma[n]$

$$\frac{d}{dz}\left\{H_1(z)\right\}=\frac{d}{dz}\left(\frac{z}{z-1}\right)=\frac{z-1-z}{(z-1)^2}=-\frac{1}{(z-1)^2}=\frac{z^{-2}}{\left(1-z^{-1}\right)^2}$$

$$\Rightarrow\frac{1}{\left(1-z^{-1}\right)^2}=-z^2\frac{d}{dz}\left\{H_1(z)\right\}\leftrightarrow(n+1)\sigma[n+1]$$

Sistemul este stabil în sens strict. Dacă $x[n]=0\Rightarrow y[n]=0$

Dacă $x[n]$ este cel mai simplu semnal în timp discret nenul, $x[n]=\delta[n]$ atunci $y[n]=h[n]=(n+1)\sigma[n+1]$, adică sistemul intră în oscilație.