A Second Order Statistical Analysis of the Hyperanalytic Wavelet Transform

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***Abstract*—We present a second order statistical analysis of the Hyperanalytic Wavelet Transform (HWT). The results are useful to design signal processing systems based on the wavelet theory.**

Keywords- Hyperanalytic Wavelet Transform; 2D Discrete Wavelet Transform; statistical analysis

I. INTRODUCTION

Wavelet Transforms (WT) are used to process images in many applications in communications. The 2D Discrete WT, 2D DWT has disadvantages [1, 2]: lack of shift invariance and poor directional selectivity, which can be diminished by a complex wavelet transform [2, 3]. We present a second order statistical analysis of a simple implementation of the HWT which is a WT with approximate shift-invariance at same redundancy and an enhanced directional selectivity [3]. We applied it in denoising and watermarking but we have not fully exploited its statistical properties. A particularity of HWT is the interscale dependency of coefficients. The paper is organized as follows. In Section II, we describe the HWT implementation starting from the implementation of 2D DWT, whose statistical analysis was reported in [4]. In Section III we present the second order statistical analysis of the HWT. Section IV concludes this paper.

II. HWT IMPLEMENTATION

A. 2D DWT Implementation

The main advantage of 2D DWT is its flexibility, as it inherits mother wavelets of 1D DWT. Iterations of 2D DWT algorithm use lowpass and highpass filters, m_0 and m_1, resulting in four subbands per scale, one of approximation and three of details: LH, HL and HH. DWT coefficients are denoted by ${}_fD_m^k$, with f the input image (a bivariate random signal), m the scale and k the orientation (1-LH, 2 -HL, 3-HH, 4-LL):

$$ {}_fD_m^k[n_1,p_1] = \left\langle f(\tau_1,\tau_2), \psi_{m,n_1,p_1}^k(\tau_1,\tau_2) \right\rangle, \tag{1} $$

The wavelets are real functions and are factorized as:

$$ \psi_{m,n,p}^k(\tau_1,\tau_2) = \alpha_{m,n}^k(\tau_1) \cdot \beta_{m,p}^k(\tau_2) \tag{2} $$

The two factors are computed using the scale function $\varphi(\tau)$ and the mother wavelets $\psi(\tau)$:

$$ \alpha_{m,n}^k(\tau) = \begin{cases} \varphi_{m,n}(\tau), & k=1,4 \\ \psi_{m,n}(\tau), & k=2,3 \end{cases}; \ \beta_{m,p}^k(\tau) = \begin{cases} \varphi_{m,p}(\tau), & k=2,4 \\ \psi_{m,p}(\tau), & k=1,3 \end{cases} \tag{3} $$

$$ \varphi_{m,n}(\tau) = 2^{-\frac{m}{2}} \varphi\left(2^{-m}\tau - n\right); \ \psi_{m,n}(\tau) = 2^{-\frac{m}{2}} \psi\left(2^{-m}\tau - n\right) \tag{4} $$

Taking into account (3)-(4) it can be written:

$$ \begin{aligned} &\psi_{m,n,p}^k(\tau_1,\tau_2) = 2^{-m}\psi^k\left(2^{-m}\tau_1 - n, 2^{-m}\tau_2 - p\right), \\ &\text{where } \psi^k(\tau_1,\tau_2) = \psi^k{}_{0,0,0}(\tau_1,\tau_2). \end{aligned} \tag{5} $$

B. Second Order Statistical Analysis of 2D DWT

The mean of the detail wavelet coefficients is zero [4]:

$$ \mu_{{}_fD_m^k} = 0, \ k = 1,2,3; \ \mu_{{}_fD_m^k} = 2^m \cdot \mu_f, \ k = 4 \tag{6} $$

where μ_f is the expected value of the input image. The intercorrelation of two wavelet coefficients in subbands k_1 and k_2 and scales m_1 and $m_2=m_1+q$ with geometrical coordinates (n_1, p_1) and (n_2, p_2) respectively, is:

$$ \begin{aligned} &R_{D1D2}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2) = 1/(4\pi^2) \\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_f(\xi_1,\xi_2) \cdot 2^{2m_1} \cdot 2^q \cdot \\ &\exp\left(-j \cdot 2^{m_1}\left(\xi_1\left(2^q n_2 - n_1\right) + \xi_2\left(2^q p_2 - p_1\right)\right)\right) \cdot \\ &\cdot \mathcal{F}\left\{\psi^{k_2}\right\}\left(2^{m_1} \cdot 2^q \xi_1, 2^{m_1} \cdot 2^q \xi_2\right) \cdot \\ &\mathcal{F}^*\left\{\psi^{k_1}\right\}\left(2^{m_1} \cdot 2^q \xi_1, 2^{m_1} \cdot 2^q \xi_2\right) d\xi_1 d\xi_2 \end{aligned} \tag{7} $$

where $S_f(\xi_1,\xi_2)$ is the power spectral density of the input image and is the two-dimensional Fourier transform.
The **inter-scale** and **inter-band dependency** ($m_1 \neq m_2, k_1 \neq k_2$) of wavelet coefficients depends on the autocorrelation of the input

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image, R_f and on the intercorrelation of the mother wavelet that generate the considered subbands, $R_{\psi^{k_2}\psi^{k_1}}$.

For k_1=k_2=k, the intercorrelation of wavelet coefficients (7) becomes an **inter-scale and intra-band dependency**. If the mother wavelet ψ^k generates by translations and dilations an orthogonal basis of $L^2(R^2)$ then:

$$R_{D1D2}\left(m_1,m_2,k_1,k_2,2^q n_1'-n_2,2^q p_1'-p_2\right)= \\ =2^{2m_1+q}R_f\left(2^{m_1+q}\left(n_2-n_1'\right),2^{m_1+q}\left(p_2-p_1'\right)\right) \quad (8)$$

If the input is 2D i.i.d. white Gaussian noise ~ $\mathcal{N}(0,\ \sigma_w^2)$, *the wavelet coefficients with different resolutions are not correlated inside a subband*.

For m_1=m_2=m the intercorrelation of wavelet coefficients becomes an **intra-scale and intra-band dependency**:

$$R_D\left(m,k,n_1-n_2,p_1-p_2\right)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S_f\left(2^{-m}\xi_1,2^{-m}\xi_2\right)\cdot \\ \cdot\exp\left(-j\left(\xi_1\left(n_2-n_1\right)+\xi_2\left(p_2-p_1\right)\right)\right)d\xi_1 d\xi_2 \quad (9)$$

At the limit for $m\rightarrow\infty$ in (9):

$$R_D\left(\infty,k,n_1-n_2,p_1-p_2\right)=S_f\left(0,0\right)\cdot\delta\left[n_2-n_1,p_2-p_1\right]. \quad (10)$$

Asymptotically, 2D DWT transforms a colored noise into a white one. Hence it is a *whitening system in an intra-band and intra-scale scenario*. A similar result is obtained for 2D i.i.d. white Gaussian noise ~ $\mathcal{N}(0,\ \sigma_w^2)$, $f(\tau_1,\tau_2)=w(\tau_1,\tau_2)$:

$$R_{Dw}\left(n_1-n_2,p_1-p_2\right)=\sigma_w^2\cdot\delta\left[n_2-n_1,p_2-p_1\right]. \quad (11)$$

In the same band and at the same scale, the 2D DWT does not correlate the i.i.d. white Gaussian noise.

C. HWT Implementation

The hypercomplex mother wavelet associated to a real mother wavelet $\psi(x,y)$ is:

$$\psi_a\left(x,y\right)=\psi\left(x,y\right)+i\mathcal{H}_x\left\{\psi\left(x,y\right)\right\}+ \\ +j\mathcal{H}_y\left\{\psi\left(x,y\right)\right\}+k\mathcal{H}_x\left\{\mathcal{H}_y\left\{\psi\left(x,y\right)\right\}\right\} \quad (12)$$

where $i^2=j^2=-k^2=-1$, and $ij=ji=k$, [5]. The HWT of the image $f(x,y)$ is:

$$HWT\left\{f\left(x,y\right)\right\}=\left\langle f\left(x,y\right),\psi_a\left(x,y\right)\right\rangle. \quad (13)$$

The 2D-HWT of the image $f(x,y)$ can be computed using the 2D-DWT of its associated hypercomplex image:

$$HWT\left\{f\left(x,y\right)\right\}=DWT\left\{f\left(x,y\right)\right\}+ \\ iDWT\left\{\mathcal{H}_x\left\{f\left(x,y\right)\right\}\right\}+jDWT\left\{\mathcal{H}_y\left\{f\left(x,y\right)\right\}\right\}+ \\ +kDWT\left\{\mathcal{H}_y\left\{\mathcal{H}_x\left\{f\left(x,y\right)\right\}\right\}\right\}= \\ \left\langle f_a\left(x,y\right),\psi\left(x,y\right)\right\rangle=DWT\left\{f_a\left(x,y\right)\right\}. \quad (14)$$

HWT uses four trees, each implemented by 2D-DWT, being adequate to a multi-wavelet environment, see [3] for more details. $\mathcal{H}_x$ is the Hilbert transform computed across lines and $\mathcal{H}_y$ across columns. The HWT coefficients are organized in two sequences of complex coefficients separated by the sign of their preferential orientation, with 6 subbands, 3 of positive orientations and 3 of negative orientations ±atan(1/2), ±π/4 and ±atan(2):

$$z_\pm=z_{\pm r}+jz_{\pm i} \\ =\left({}_fD^{1,2,3}\mp{}_{\mathcal{H}_y\{\mathcal{H}_x\{f\}\}}D^{1,2,3}\right)+j\left({}_{\mathcal{H}_x}D^{1,2,3}\pm{}_{\mathcal{H}_y}D^{1,2,3}\right). \quad (15)$$

III. HWT Statistical Analysis

The expectation of coefficients z_+ and z_- is:

$$E\left\{z_+\right\}=E\left\{{}_fD^{1,2,3}\right\}-E\left\{{}_{\mathcal{H}y\{\mathcal{H}x\{f\}\}}D^{1,2,3}\right\}+ \\ jE\left\{{}_{\mathcal{H}x}D^{1,2,3}\right\}+jE\left\{{}_{\mathcal{H}y}D^{1,2,3}\right\}=0=E\left\{z_-\right\} \quad (16)$$

A. Intercorrelation between Real and Imaginary Parts of the Coefficients in Subbands with Same Type of Orientation

Applying the definition of the statistical correlation for the real and imaginary parts of the coefficients z_+ we obtain a sum of correlations of 2D DWT coefficients:

$$R_{z_{+r}z_{+i}}\left[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2\right]= \\ =R_{D_fD_{\mathcal{H}x}}\left(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2\right)- \\ -R_{D_{\mathcal{H}y\{\mathcal{H}x\}}D_{\mathcal{H}x}}\left(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2\right)+ \\ +R_{D_fD_{\mathcal{H}y}}\left(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2\right)- \\ -R_{D_{\mathcal{H}y\{\mathcal{H}x\}}D_{\mathcal{H}y}}\left(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2\right). \quad (17)$$

Each term of the right hand side can be computed using (7). In (7), instead of S_f we must substitute the power spectral densities and interspectra: $S_{f\mathcal{H}x}$, $S_{\mathcal{H}y\{\mathcal{H}x\}\mathcal{H}x}$, $S_{f\mathcal{H}y}$ and $S_{\mathcal{H}y\{\mathcal{H}x\}\mathcal{H}y}$. These are:

$$
\begin{aligned}
&S_{f\mathcal{H}x}(\xi_1,\xi_2) = j\,\mathrm{sgn}\,\xi_1 S_f(\xi_1,\xi_2),\\
&S_{\mathcal{H}y\{\mathcal{H}x\}\mathcal{H}x}(\xi_1,\xi_2) = -j\,\mathrm{sgn}^2\,\xi_1\,\mathrm{sgn}\,\xi_2 S_f\\
&S_{f\mathcal{H}y}(\xi_1,\xi_2) = j\,\mathrm{sgn}\,\xi_2 S_f(\xi_1,\xi_2),\\
&S_{\mathcal{H}y\{\mathcal{H}x\}\mathcal{H}y}(\xi_1,\xi_2) = -j\,\mathrm{sgn}\,\xi_1\,\mathrm{sgn}^2\,\xi_2 S_f.
\end{aligned} \tag{18}
$$

and after their substitution, the equation (17) becomes:

$$
\begin{aligned}
&R_{z_{+r}z_{+i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=\\
&j/(4\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)+\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_1\big)\cdot\\
&\cdot\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)+\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)+\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)\cdot\\
&\cdot\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_2\big)\Big]\cdot\cdot S_f\big(2^{-m_1}2^{-q}\nu_1,2^{-m_1}2^{-q}\nu_2\big)\cdot\\
&\cdot 2^{-q}\cdot e^{-j\left(\nu_1\left(n_2-2^{-q}n_1\right)+\nu_2\left(p_2-2^{-q}p_1\right)\right)}\cdot\\
&\cdot\mathcal{F}\{\psi^{k_2}\}(\nu_1,\nu_2)\cdot\mathcal{F}^*\{\psi^{k_1}\}(\nu_1,\nu_2)\,d\nu_1 d\nu_2.
\end{aligned} \tag{19}
$$

The intercorrelation of real and imaginary parts of z_- is given in equation (20). At the limit for $m_1\to\infty$ in equations (19) and (20) we obtain the result in equation (21) because sgn(0)=0. This result proves that real and imaginary parts of z_+ and z_- are asymptotically decorrelated in an inter-band context.

$$
\begin{aligned}
&R_{z_{-r}z_{-i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=\\
&j/(4\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)-\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_1\big)\\
&\cdot\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)-\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)+\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)\cdot\\
&\cdot\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_2\big)\Big]\cdot S_f\big(2^{-m_1}2^{-q}\nu_1,2^{-m_1}2^{-q}\nu_2\big)\\
&2^{-q}\cdot e^{-j\left(\nu_1\left(n_2-2^{-q}n_1\right)+\nu_2\left(p_2-2^{-q}p_1\right)\right)}\cdot\\
&\cdot\mathcal{F}\{\psi^{k_2}\}(\nu_1,\nu_2)\cdot\mathcal{F}^*\{\psi^{k_1}\}(\nu_1,\nu_2)\,d\nu_1 d\nu_2.
\end{aligned} \tag{20}
$$

$$
\lim_{m_1\to\infty} R_{z_{\pm r}z_{\pm i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=0 \tag{21}
$$

B. Correlation of Real Parts and of Imaginary Parts of Coefficients in Subbands with Same Type of Orientation

We identify inter-band and inter-scale dependencies of the real and respectively of the imaginary parts of z_+ and z_-:

$$
\begin{aligned}
&R_{z_{+i}z_{+i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=\\
&=R_{D\mathcal{H}xD\mathcal{H}x}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)+\\
&+R_{D\mathcal{H}xD\mathcal{H}y}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)+\\
&+R_{D\mathcal{H}yD\mathcal{H}x}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)+\\
&R_{D\mathcal{H}yD\mathcal{H}y}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2).
\end{aligned} \tag{22}
$$

Each term in the right hand side can be computed using (7), but instead of S_f we must substitute with $S_{\mathcal{H}x}$, $S_{\mathcal{H}x\mathcal{H}y}$, $S_{\mathcal{H}y\mathcal{H}x}$ and $S_{\mathcal{H}y}$. We have similar relations for other orientations, for real part of z_+ and z_-. The equation (22) becomes:

$$
\begin{aligned}
&R_{z_{+i}z_{+i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=1/(4\pi^2)\cdot\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_1\big)+2\,\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)\\
&\cdot\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)+\mathrm{sgn}^2\big(2^{-m_1}2^{-q}\nu_2\big)\Big]\cdot\\
&2^{-q}\cdot e^{-j\left(\nu_1\left(n_2-2^{-q}n_1\right)+\nu_2\left(p_2-2^{-q}p_1\right)\right)}\cdot\\
&\cdot\mathcal{F}\{\psi^{k_2}\}(\nu_1,\nu_2)\cdot\mathcal{F}^*\{\psi^{k_1}\}(\nu_1,\nu_2)\,d\nu_1 d\nu_2
\end{aligned} \tag{23}
$$

At the limit for $m_1\to\infty$, the right hand side of the last equation becomes equal with zero because sgn(0)=0.

The imaginary parts of the coefficients z_+ are asymptotically decorrelated [6].

C. Intercorrelation of Coefficients in Subbands with Opposite Type of Orientation

The HWT subbands are positive oriented for z_+ and negative orientated for z_-. The intercorrelation between the imaginary parts of the coefficients z_+ and z_- is:

$$
\begin{aligned}
&R_{z_{+i}z_{-i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=\\
&=R_{D\mathcal{H}xD\mathcal{H}x}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)-\\
&-R_{D\mathcal{H}xD\mathcal{H}y}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)+\\
&+R_{D\mathcal{H}yD\mathcal{H}x}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2)-\\
&-R_{D\mathcal{H}yD\mathcal{H}y}(m_1,m_2,k_1,k_2,n_1-n_2,p_1-p_2).
\end{aligned} \tag{24}
$$

Each term of the right hand side of the last equation can be computed using (7) by the substitution of S_f with $S_{\mathcal{H}x}$, $S_{\mathcal{H}x\mathcal{H}y}$, $S_{\mathcal{H}y\mathcal{H}x}$ and $S_{\mathcal{H}y}$ obtaining:

$$
\begin{aligned}
&R_{z_{+i}z_{-i}}[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2]=1/(4\pi^2)\cdot\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\big|\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_1\big)\big|^2-\big|\mathrm{sgn}\big(2^{-m_1}2^{-q}\nu_2\big)\big|^2\Big]\cdot\\
&S_f\big(2^{-m_1}2^{-q}\nu_1,2^{-m_1}2^{-q}\nu_2\big)\cdot\\
&2^{-q}\cdot e^{-j\left(\nu_1\left(n_2-2^{-q}n_1\right)+\nu_2\left(p_2-2^{-q}p_1\right)\right)}\cdot\\
&\cdot\mathcal{F}\{\psi^{k_2}\}(\nu_1,\nu_2)\cdot\mathcal{F}^*\{\psi^{k_1}\}(\nu_1,\nu_2)\,d\nu_1 d\nu_2
\end{aligned} \tag{25}
$$

The first factor under the integral from the right hand side of the last equation equals zero for any pair of not nulls real numbers (ν_1,ν_2), reason for which it can be written:

$$R_{z_{+i}z_{-i}}\left[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2\right]=0 \text{ a.e.w}, \forall \nu_{1,2}\neq 0 \quad (26)$$

Imaginary parts of positively oriented subbands are not correlated with imaginary parts of the negatively oriented subbands for any finite values m_1 and m_2. This is a more general result than former ones which are of asymptotical nature only. The same value is obtained for intercorrelation between real parts of z_- and imaginary parts of z_+, or for intercorrelation between real parts of z_+ and z_-:

$$R_{z_{\mp r}z_{\pm i}}\left[m_1,m_2,k_1,k_2,n_1,n_2,p_1,p_2\right]=0 \text{ a.e.w} \quad (27)$$

D. Inter-scale and Intra-band Dependencies

For $k_1=k_2=k$, if the mother wavelet ψ^k generates by translations and dilations an orthogonal basis of $L^2(R^2)$ then:

$$\begin{aligned}&R_{z_{+i}z_{+i}}\left[m_1,m_2,k,2^q n_1'-n_2,2^q p_1'-p_2\right]=\\&=2^{2m_1+q}\Big\{R_{\mathcal{H}x\mathcal{H}x}\left(2^{m_1+q}(n_2-n_1'),2^{m_1+q}(p_2-p_1')\right)+\\&+R_{\mathcal{H}x\mathcal{H}y}\left(2^{m_1+q}(n_2-n_1'),2^{m_1+q}(p_2-p_1')\right)+\\&R_{\mathcal{H}y\mathcal{H}x}\left(2^{m_1+q}(n_2-n_1'),2^{m_1+q}(p_2-p_1')\right)-\\&+R_{\mathcal{H}y\mathcal{H}y}\left(2^{m_1+q}(n_2-n_1'),2^{m_1+q}(p_2-p_1')\right)\Big\}\end{aligned} \quad (28)$$

Similar results are obtained for $R_{z_{+r}z_{+r}}$, $R_{z_{-i}z_{-i}}$ and $R_{z_{-r}z_{-r}}$. In an inter-scale and intra-band context, correlation functions of the HWT coefficients depend solely on the correlations of the four input images f, $\mathcal{H}_x\{f\}$, $\mathcal{H}_y\{f\}$, $\mathcal{H}_y\{\mathcal{H}_x\{f\}\}$, if orthogonal wavelets are used.

E. Intra-scale and Intra-band Dependencies

For $m_1=m_2=m$ we have:

$$\begin{aligned}&R_{z_{+i}z_{+i}}\left[m,k,n_1'-n_2,p_1'-p_2\right]=\\&=2^{2m}\Big\{R_{\mathcal{H}x\mathcal{H}x}\left(2^{m}(n_2-n_1'),2^{m}(p_2-p_1')\right)+\\&+R_{\mathcal{H}x\mathcal{H}y}\left(2^{m}(n_2-n_1'),2^{m}(p_2-p_1')\right)+\\&+R_{\mathcal{H}y\mathcal{H}x}\left(2^{m}(n_2-n_1'),2^{m}(p_2-p_1')\right)+\\&+R_{\mathcal{H}y\mathcal{H}y}\left(2^{m}(n_2-n_1'),2^{m}(p_2-p_1')\right)\Big\}\end{aligned} \quad (29)$$

Using Wiener-Hincin theorem we obtain equation (30). At the limit for $m\to\infty$, the equation (30) becomes (31) which represent the autocorrelation of a white noise. Similar asymptotic results are obtained for the subbands z_{+r}, z_{-i} and z_{-r}. HWT can also be seen as a whitening system in an intra-scale and intra-band scenario, just like the 2D DWT.

$$\begin{aligned}&R_{z_{+i}z_{+i}}\left[m,k,n_1-n_2,p_1-p_2\right]=1/(4\pi^2)\cdot\\&\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[S_{\mathcal{H}x\mathcal{H}x}\left(2^{-m}\xi_1,2^{-m}\xi_2\right)+S_{\mathcal{H}x\mathcal{H}y}\left(2^{-m}\xi_1,2^{-m}\xi_2\right)\\&+S_{\mathcal{H}y\mathcal{H}x}\left(2^{-m}\xi_1,2^{-m}\xi_2\right)+S_{\mathcal{H}y\mathcal{H}y}\left(2^{-m}\xi_1,2^{-m}\xi_2\right)\Big]\cdot\\&\cdot\exp\left\{-j\left[\xi_1(n_2-n_1)+\xi_2(p_2-p_1)\right]\right\}d\xi_1 d\xi_2\end{aligned} \quad (30)$$

$$\begin{aligned}&R_{z_{+i}z_{+i}}\left[\infty,k,n_1-n_2,p_1-p_2\right]=\left(S_{\mathcal{H}x\mathcal{H}x}(0,0)+S_{\mathcal{H}x\mathcal{H}y}(0,0)\right.\\&\left.+S_{\mathcal{H}y\mathcal{H}x}(0,0)+S_{\mathcal{H}y\mathcal{H}y}(0,0)\right)\cdot\delta\left[n_2-n_1,p_2-p_1\right]\end{aligned} \quad (31)$$

IV. CONCLUSION

We generalized the second order statistical analysis of 2D DWT from [4] for HWT. This WT seems more complicated than 2D DWT, because of the greater number of subbands and complex coefficients. HWT coefficients have strong inter-scale and inter-band dependencies. Real and imaginary parts of coefficients in subbands with same type of orientation are asymptotically decorrelated. In an inter-band and inter-scale context, real and respectively imaginary parts of z_+ and z_- are asymptotically decorrelated. We analyzed coefficients in subbands with opposite type of orientation. Intercorrelations are zero a.e.w. even for finite number of scales. This allows parallel processing of the HWT coefficients in subbands with opposite type of orientation [7]. HWT coefficients correlations are independent of the mother wavelets in an inter-scale and intra-band context, depending on correlations of the four input images f, $\mathcal{H}_x\{f\}$, $\mathcal{H}_y\{f\}$, $\mathcal{H}_y\{\mathcal{H}_x\{f\}\}$ only, if orthogonal wavelets are used. HWT is a whitening system in an intra-scale and intra-band scenario, similarly to 2D DWT. We analyzed the two WTs in only three scenarios: inter-scale and inter-band, inter-scale and intra-band and intra-scale and inter-band. The 2D DWT and the HWT have similar statistical behaviors in inter-scale scenarios. Asymptotically, HWT has higher decorrelation strength in intra-band scenarios.

REFERENCES

[1] N. Kingsbury, "Complex Wavelets for Shift Invariant Analysis and Filtering of Signals," Applied Comp. Harm. Anal. 10, 2001, 234-253.

[2] N. G. Kingsbury, "A Dual-Tree Complex Wavelet Transform with improved orthogonality and symmetry properties," Proc. IEEE Conf. Image Processing, Vancouver, September 11-13, 2000, paper 1429.

[3] I. Adam, C. Nafornita, J.-M. Boucher and A. Isar, "A New Implementation of the Hyperanalytic Wavelet Transform," Proc. IEEE Sympo. ISSCS 2007, Iasi, Romania, 401-404.

[4] C. Nafornita, I. Firoiu, D. Isar, J.-M. Boucher, A. Isar, "A Second Order Statistical Analysis of the 2D DWT," Proc. IEEE Conf Comm 2010, Bucharest, Romania, June 2010.

[5] C. Davenport, Commutative Hypercomplex Mathematics, http://home.usit.net/~cmdaven/cmdaven1.htm.

[6] I. Firoiu, C. Nafornita, D. Isar, J.-M. Boucher, A. Isar, "An Asymptotic Statistical Analysis of the Hyperanalytic Wavelet Transform," European Conf. Circuits & Systems for Communications ECCSC 2010, Nov. 23–25, 2010, Belgrade, Serbia, in press.

[7] I. Firoiu, C. Nafornita, J.-M. Boucher, A. Isar, "Image Denoising Using a New Implementation of the Hyperanalytic Wavelet Transform," IEEE Trans. on Instrum & Meas, Vol. 58, Issue 8, Aug 2009, pp. 2410-2416.